Spatial Distribution of Flows in Transportation Networks. A Model Based on Bounded Rationality Assumption

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This paper refers to a Dynamic Traffic Assignment Problem. A consecutive dynamic model of traffic flows is formulated. Some of its dynamical properties (including existence of chaotic solutions and bifurcations) are examined in two special cases.

Keywords: transportation networks, dynamic traffic assignment problem, model of traffic flows.

1. INTRODUCTION

Theoretical analysis of the distribution of streams in the transport network plays an important role in the theory and practice of transport. The natural alternative for these analyzes are empirical studies, but they are usually very difficult and expensive when hold in the macro-scale. It follows the natural need to construct and apply the sufficiently accurate models describing spatial distribution of flows in various kinds of transport modes and networks.

This paper concerns the dynamic aspects of traffic assignment problem, focusing on the nature of the network flow adjustment process which is a consequence of carriers decisions and actions in their search of better routes. The main purpose of this study is to formulate the model based on the assumption about carriers' bounded rationality and examine some special cases. The terminology used later is dedicated to freight transport, but the results of analysis (including formal derivation of dynamic equations) may be directly adopted to the case of passenger transport.

It seems that most of past studies have used the deterministic models, but some of them have based on the probabilistic paradigm. These studies were used to examine the stability of different types of network equilibria (first of all the *Deterministic User Equlibrium*, and its stochastic variant, see for example Florian et all., [7]). The results strongly depend on the „nature of time" in models. Dynamics analysis for discrete time case one can find in the papers of Smith [15], Friesz [8], Watling [17,18], Zhang and Nagurney [19], Cho and Hwang [5], Mounce [14] and others. Analysis of discrete time models can be found for example in Cantarella and Casscetta [3] or Bie and Lo [2]. It seems that all these models base on the fundamental assumption of rational behavior of market participants (i.e. users of a network).

In this paper we partly weaken this assumption in favor of the concept of bounded (namely procedural) rationality. This machinery allows us to propose new equations for estimating the spatial distribution of flows. These equations will be used to identify the basics of attraction of equilibrium states and also may be used for verifying the stability of networks equilibria.

2. THE MODEL

The structure of transport network, information about the structure of population of transport companies (hereinafter referred to as *carriers*) and assumptions about their decision-making mechanism (especially in terms of choice of transport routes) play an important role in the spatial distribution modeling of flows in the network. We briefly discuss these categories.

Carriers

In many countries, including Poland, road freight transport is decentralized. Transport services are provided by a relatively large number of small carriers [1]. There are significant differences in estimations of the number of carries in Poland. Burnewicz [3] estimated it at over 80 thousand (in 2006). Other authors (Bentkowska-Senator, Kordel, [1]), basing on various kinds of sources, estimate it at about 110 thousand, including about 51 thousand national transport companies and about 44 thousand of commercial ones. It is worth mentioning that according to Polish Statistical Office data (GUS, [16]), at the end of the year 2008 there were about 1200 companies with 11-50 lorries and a few more than 100 companies with a larger fleet. Despite of the various kinds of changes (registrations of new companies, re-registration of the others including cases of bankruptcy), it seems that their total number remains approximately constant over time. Consequently, there are some reasons to treat the set of carriers as homogeneous population consisting of small market players (i.e. the activity of a single company has a negligible impact on the total volume of traffic). It also seems to be an acceptable assumption about the independence of different carriers, which means that they choose their routes (between the given origin and destination place) independently to each other.

Flows in a network

We consider a *transport network* (network for short) consisting of directed multigraph. Let W denote the set of all pair of nodes. For given $i \in W$ let P_j be a set of acyclic paths (routes) connecting initial and ending nodes of i. A path flow assignment (a spatial distribution of path flows) is represented by a vector $h=(h_n)$, $p \in P= {}_{i \in W} P_i$, which components describe the quantities of flows on a consecutive paths. Path flows uniquely determine arc flows, v. There is a linear relationship between these vectors, namely $v = \delta h$, where δ is the incidence matrix of paths and arcs. Every path $p \in \mathbf{P}$ is characterized by a *cost*, c_p, of travel along p. We will assume that c_p depends on h and the function $c_p = c_p(h)$ is known to every carrier.

3. THE DYNAMICS OF FLOWS ASSIGNMENT

The volume of total traffic is a result of aggregation of unit flows generated by particular carriers. Hence, the spatial distribution of flows reflects carriers' decisions on the selection of transport routes. In particular, the size $h_{p,t}$ of flow in time t on the path p is given by

$$h_{p,t} = \pi_{p,t} D_{i,t} \forall\ p \in \mathbf{P}_i,\ i \in W, \tag{1}$$

where D_i is the total demand for transport (demand for short) between pair $i \in W$ of nodes, $\pi_{p,t}$ is the probability of choosing path p by a representative carrier. The immediate consequence of (1) are the formulas for flows' increments on the path $p \in \mathbf{P}_i$ ($i \in W$):

$$dh_{p,t} = D_{i,t} d\pi_{p,t} + \pi_{p,t} dD_{i,t}, \tag{2}$$

$$\Delta h_{p,t} = \Delta D_{i,t} \pi_{p,t+1} + D_{i,t} \Delta \pi_{p,t} \tag{3}$$

for the case of continuous and discrete time respectively. Therefore derivation of the specific form of equation dynamics requires specification of changes of two factors: global demand and the probabilities of road selections. The first one is determined by macroeconomic factors, in particular changes in the volume production in various sectors of the economy. The method of modeling the changes of the second factor - probabilities $\pi_{p,t}$ - may be based on the concept of *procedure* (originally taken from the population game theory). In the case of modeling, the behaviour of carriers the procedure may be regarded as an algorithm for defining „the sufficiently good" transport strategy for currently observed state of the network. The results of the procedure can also be used to verify that previously applied strategy remains acceptable. The typical procedure may consist of making additional observation, taking into account some additional data, carrying out additional specific

[1] Polish law (Ustawa o swobodzie działalności gospodarczej, Dz.U. Nr 173, poz. 1807 z późn. zm.) states that an entrepreneur is a person, legal person or agency having legal capacity in pursuing the business. In addition Motor Transport Act (Ustawa o transporcie samochodowym, Dz.U. z 2004r., Nr 204, poz. 288) states the road transport entrepreneur as a person who is allowed to carry out business activities in the field of road transport.

calculations, etc. We will assume that the decision-making mechanism of every carrier is consistent with the Calvo scheme: the carrier makes the procedure and may implement its results to re-optimize and improve his transport strategy, or keep the status quo.

Accordingly to the last assumption, let $\lambda_p(t,t+\varepsilon)$ (for fixed $\varepsilon > 0$) be the probability that a carrier, who just before the moment t was willing to choose the path $p \in \mathbf{P}_i$, also chooses this path at time $t+\varepsilon$. For a fixed pair of nodes $i \in W$ the probabilities $(\pi^{\mathbf{proc}}_{p,t} : k \in \mathbf{P}_i)$ satisfy the following equation:

$$\pi_{p,t+\varepsilon} = \pi^{\mathbf{proc}}_{p,t}\lambda_p(t,t+\varepsilon) + \pi_{p,t-0}(1-\lambda_p(t,t+\varepsilon)). \tag{4}$$

For discrete time model ε can be set to 1. For continuous time case, assuming that the probabilities depend almost linearly on the length of time period $[t,t+\varepsilon]$ (ie. there exists a constant $\lambda_{p,t} > 0$ such that $\lambda_p(t,t+\varepsilon) = \lambda_p\varepsilon + o(\varepsilon)$ if $\varepsilon \downarrow 0$), it can be easily seen that (pra1b) follows the existence of limit

$$\lim_{\varepsilon \downarrow 0}(\pi_{p,t+\varepsilon} - \pi_{p,t-0})/\varepsilon = \lambda_{p,t}(\pi^{proc}_{p,t} - \pi_{p,t-0}) \tag{5}$$

and consequently continuity and differentiability of the map $t \mapsto \pi_{p,t}$. The rule of probability changes may be written as:

$$\pi_{p,t+1} - \pi_{p,t} = \lambda_{p,t}(\pi^{\mathbf{proc}}_{p,t} - \pi_{p,t}) \tag{6}$$

$$d\pi_{p,t} = \lambda_{p,t}\left(\pi^{\mathbf{proc}}_{p,t} - \pi_{p,t}\right)dt \tag{7}$$

in discrete and continuous time model respectively.

Finally, the equation describing the dynamics of flows on the path $p \in \mathbf{P}_i$ $(i \in W)$ has the following form (for discrete and continuous time case respectively):

$$\begin{aligned} h_{p,t+1} - h_{p,t} &\\ &= (\pi^{\mathbf{proc}}_{p,t}\lambda_{p,t} + \pi_{p,t}(1-\lambda_{p,t}))(D_{i,t+1} - D_{i,t}) \\ &\quad + \lambda_{p,t}(\pi^{\mathbf{proc}}_{p,t} - \pi_{p,t})D_{i,t}, \end{aligned} \tag{8}$$

$$dh_{p,t} = \lambda_{p,t}\left(\pi^{\mathbf{proc}}_{p,t} - \pi_{p,t}\right)D_{i,t}dt + \lambda_{p,t}\pi_{p,t}dD_{i,t}. \tag{9}$$

Critical points of this system define a stationary procedural equilibrium. Performing the procedure by each of the carrier and possible use of its results (with probability λ) does not change the distribution of flows in the network[2]. It is natural to assume that probabilities $(\pi^{\mathbf{proc}}_{p,t} : p \in \mathbf{P})$ depend on previously observed transport costs. A quite general model for those probabilities (for discrete and continuous time case) may have the following form:

$$\pi^{\mathbf{proc}}_{p,t} = \begin{cases} A_i \sum\limits_{s \geq 0} w_{t-s}\phi(c_p(h_{t-s}) - c_i^0(h_{t-s})), \\ A_i \int\limits_0^{\infty} w(t-s)\phi(c_p(h_{t-s}) - c_i^0(h_{t-s}))ds, \end{cases} \tag{10}$$

for every $p \in \mathbf{P}_i$, $i \in W$, where $c_i^0 = \min\{c_k : k \in \mathbf{P}_i\}$ is a minimal cost of delivery between the pair $i \in W$, non-negative functions ϕ and w are given. First of them is non-increasing, the second one - non-decreasing. Functions A_i $(i \in W)$ guarantee that (10) defines the probability distributions, i.e. $\sum_{k \in \mathbf{P}_i} \pi^{\mathbf{proc}}_{k,t} = 1$ for every pair $i \in W$.

According to the above remarks, the dynamic properties of the model depend on the variability of aggregate demand, definition of the procedure and the tendency of carriers to accept and take account of its results. These remarks may be illustrated by the following examples.

4. TWO EXAMPLES

We solve the dynamics equations for a graph consisting of two nodes connected and two arcs (equivalently paths $p = 1, 2$), with the following

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[2]It can be shown (see for example [2], [6]) that under some assumptions, the fixed (critical) points of (8,9) corresponds to the stochastic user equilibrium states.

unit costs of transportation: $c_1(h) = h_1$, $c_2(h) = 2h_2$. The phase space consisting of all admissible distributions of flows is the simplex defined by the conditions $x_1 + x_2 = D$ and $x_1, x_2 \geq 0$.

We shall additionally assume that the population of carriers is homogeneous and the probabilities of making procedures do not vary over time, $\lambda_t = \lambda$. Their common value can be regarded as a measure of carriers' tendency to makie new procedures ie. to re-optimize their behaviour. The procedures have Markov property, i.e. their results depend on the actually observed flows and do not depend on the former states. Speaking more formally, this lack of memory corresponds to the adoption of $w_s = 0$ (for $s \neq 0$) and $w(s) = \delta(s)$ (Dirac delta) in the formula (10) (in the case of discrete and continuous time model respectively). This leads to the following formula for probabilities of choosing paths after making the procedure:

$$\pi_{p,t}^{\mathbf{proc}} = A\phi(c_p - c^0), \tag{11}$$

where $p = 1,2$ and $c^0 = \min(c_1, c_2)$. In further examples we shall consider the special case with exponential dependence in (11):

$$\pi_{p,t}^{\mathbf{proc}} \sim \exp(-\alpha(c_p - c^0)) \sim \exp(-\alpha c_p), \tag{12}$$

where $\alpha > 0$ specifies the sensitivity of $\pi_{p,t}^{\mathbf{proc}}$ on the changes of the transportation cost. Normalization of (eq:11 additional) leads to the following multinomial logic formulas for $p = 1,2$:

$$\pi_{p,t}^{\mathbf{proc}} = \exp(-\alpha c_p(h_p))\big((\exp(-\alpha c_1(h_1)) + \exp(-\alpha c_2(h_2))\big). \tag{13}$$

The graph of (13) as a function of h_1 is shown in Fig. 1.

In next two examples, we will examine some properties of the dynamics of flows. Due to the complicated nature of the equations, even making a qualitative analysis of phase portrait requires application of some numerical procedures. The presented results have been obtained using the IDMC (Interactive Dynamical Model Calculator, see for example [11,12]) and from author's scripts written for **R** software.

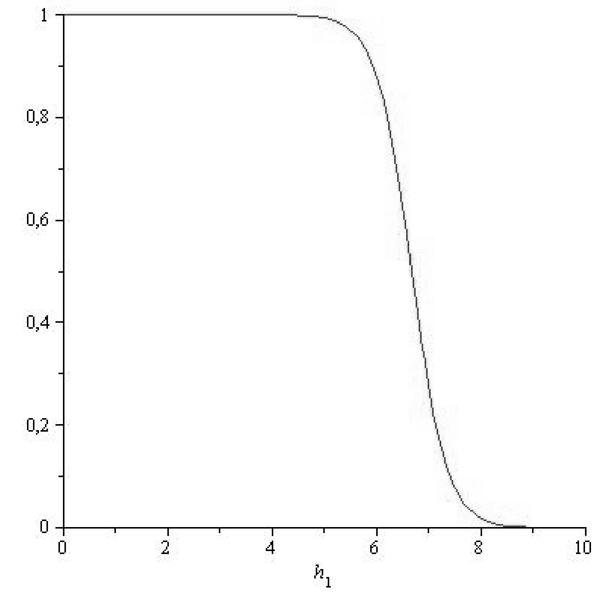

Fig. 1. The graph of (13) for $\alpha = 1$ and $D = 10 = h_1 + h_2$

Example 1.

Let us assume that the total demand is constant ($D_t = D$). In this case the formulas (8,9) have the following form:

$$h_{p,t+1} - h_{p,t} = \lambda\left(\frac{D\exp(-\alpha c_p(h_{p,t}))}{\exp(-\alpha c_1(h_{1,t})) + \exp(-\alpha c_2(h_{2,t}))} - h_{p,t}\right), \tag{14}$$

$$dh_{p,t} = \lambda\left(\frac{D\exp(-\alpha c_p(h_{p,t}))}{\exp(-\alpha c_1(h_{1,t})) + \exp(-\alpha c_2(h_{2,t}))} - h_{p,t}\right)dt. \tag{15}$$

It is easy to check that the derivative (with respect to h_p) of the right hand side of both last formulas is locally bounded. This fact guarantees existence and uniqueness of solutions of initial problems for (14,15).

Qualitative properties of the phase portrait depend significantly on the value of model parameters. For small demand, the results of procedures does not have a significant impact on the flows' distribution. Consequently, the dynamics of the system is relatively conservative and there is exactly one asymptotically stable equilibrium point for all values of $\lambda \in [0,1]$. As the demand grows the range of parameter λ for which there is only one equilibrium point shrinks. For sufficiently large values of λ periodic solutions (with periods ≤ 2) and quasi-periodic ones occur. For larger values of α the dynamics is even more chaotic. For example, if $\alpha = 3$, there are some ranges of λ where the dynamics is chaotic (see Fig. 2 and positive values of estimated Lyapunov exponents, Fig. 3). In practice, however, the value of λ appears to be relatively small (although verification of this claim certainly requires a detailed study), so one does not expect a chaotic change of flows in a network (see Fig. 4). In the case of continuous time there is no chaotic dynamics in the network considered in this example. Depending on the parameter values one can observe stable critical points, periodic solutions, or - most frequently - quasi-periodic solutions.

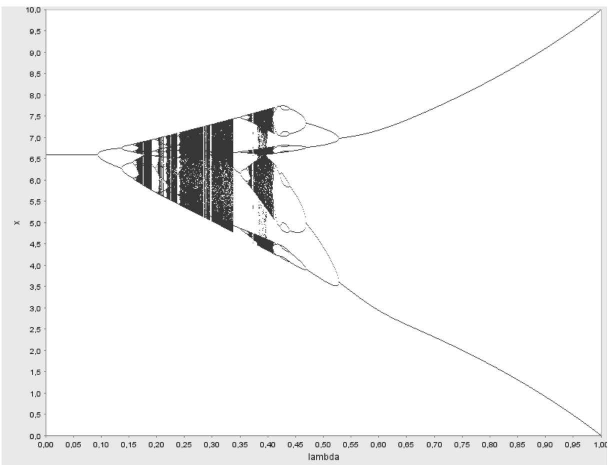

Fig. 2. Bifurcation diagrams for the parameter $\lambda \in [0,1]$ in (14). The another parameters are: $\alpha = 3$, $D = 10$ and initial state $h_1 |_{t=0} = D/2$.

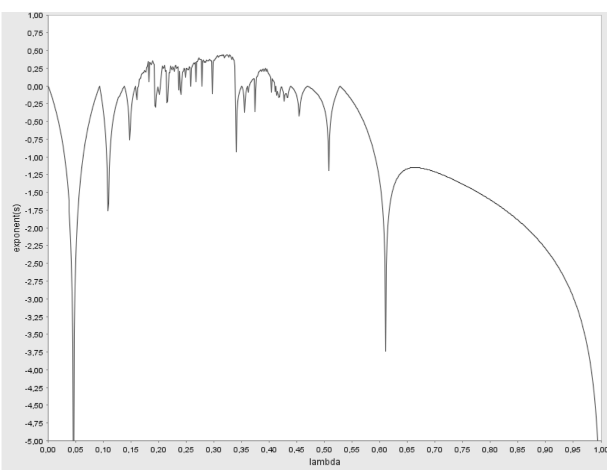

Fig. 3. Lyapunov exponents for the system (14). The values of parameters are the same as in Fig. 2.

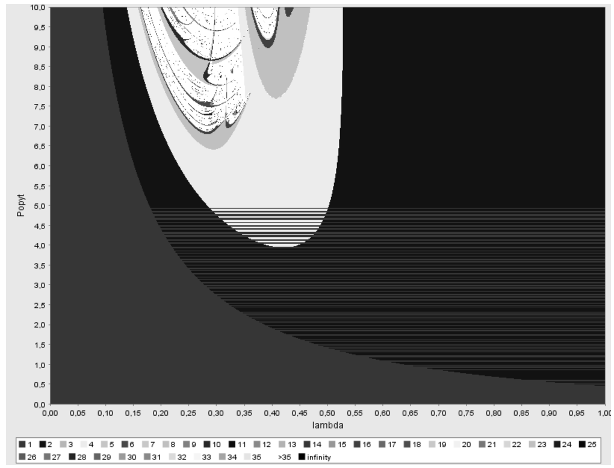

Fig. 4. Two-dimensional bifurcation diagrams for the system (14) and parameters D, λ and $\alpha = 3$.

Example 2.

We will continue solving dynamic equations in the case considered in the previous example. The only difference is that the demand may vary over time. It is obvious that the detailed description of changes in demand for transport requires quite a subtle analysis of all sectors of the economy. In the first step of approximation we can neglect the external factors and assume that the demand rate of change and its deviation from the trend depend only on the current size of demand. This leads to the following formulas for continuous and discrete time model respectively:

$$D_{t+1} - D_t = D_t\left(r(D_t) + \sigma(D_t)\varepsilon_t\right) \qquad (16)$$

$$dD_t = D_t\left(r(D_t) + \sigma(D_t)dW_t\right) \qquad (17)$$

where r, σ define the average rate of transport demand and its standard deviation, (ε_t) and (W_t) denote the white noise process and Wiener process respectively. In the simplest case, especially reasonable for short-term horizon, we can assume that the functions r, σ are constant. The aggregate demand is then defined as a process of geometric Brownian motion:

$$D_t = D_0 \prod_{\tau=0}^{t-1} (1 + r + \sigma\varepsilon_\tau),$$
$$D_t = D_0 \exp\left((r - s^2/2)t + sW_t\right).$$

The equations (8,9) and (16,17) have the following form:

$$h_{p,t+1} - h_{p,t} = \lambda D_{t+1} \frac{\exp(-\alpha c_p)}{\exp(-\alpha c_1) + \exp(-\alpha c_2)} - \lambda h_{p,t} + (1-\lambda)(1 + r + \sigma\varepsilon_t), \qquad (18)$$

$$dh_{p,t} = \left(\lambda D_t \frac{\exp(-\alpha c_p)}{\exp(-\alpha c_1) + \exp(-\alpha c_2)} + (r - \lambda) h_{p,t} \right) dt + \sigma h_{p,t} dW_t. \tag{19}$$

It is not hard to prove that for a given initial value of demand, there exists a strong solution of this equation and it is unique (i.e. solutions satisfying the same initial condition are equal almost surely). It seems to be impossible (or at least very hard) to find an analytic form of solution of (18,19). The expected value and standard deviation functions of the solution were calculated solutions using Milstein scheme (see [10]) implemented in the Stochastic Differential Equation (*sde*) package of **R** software. Sample trajectories of the solution is shown in Fig. 5, some of the estimated moments of the process $t \mapsto h_{1,t}$ is shown in Fig. 6.

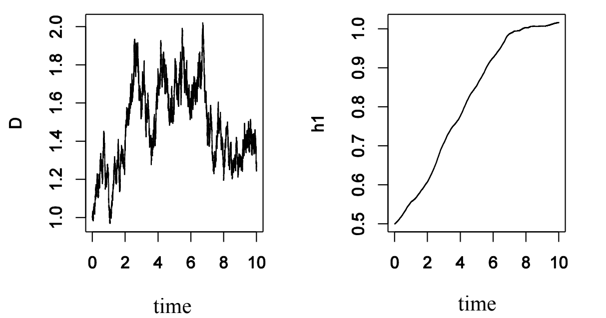

Fig. 5. A sample path of total demand (left panel) and a flow along path 1 (right panel). The parameters are equal: $\lambda = 0.1$, $r = 0.05$, $s = 0.25$. Initial state $h_1 |_{t=0} = D/2$.

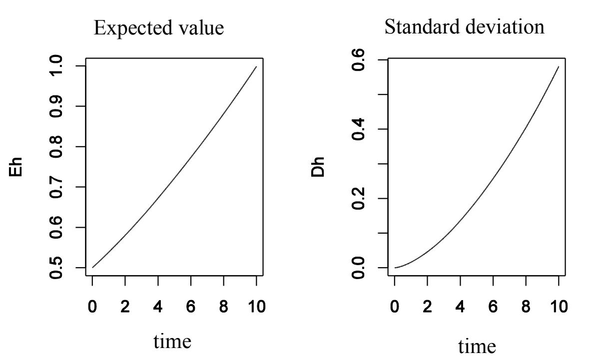

Fig. 6. The expected value and standard deviation of flow along path 1 (left and right panels respectively). Results of Monte Carlo Simulations. Values of parameters are the same as in the previous figure.

5. FINAL REMARKS

In the considered examples, the demand for transport category was exogenous. Creating a more subtle versions of the models, including long-term analysis of demand, it is necessary to model the interaction between transport and economics. It obviously tends to increase the complexity of the model, because of the difficulties in the modeling of the relationships between transport and other sectors of the economy. It seems that the model describing the quantity and the spatial distribution of demand for transport should incorporate most of macro-characteristics of economic growth, for example the structure of population, the level of income, the size of GDP, volume of foreign trade, consumption, the structure of settlement networks, price indexes, the structure of employment, the degree of market liberalization (especially in transport sector), new technology, infrastructure, etc. ([3], [13]).

Another issue of major importance is an appropriate choice of procedure (or class of them in the case of heterogeneous populations of carriers). The case of (generalized) markovian-type procedures, independently performed by different carriers, seems to play a crucial role in modeling dynamics of flows. In this case the spatial structure of flows may be (under some additional technical assumptions) obtained by solving the Chapman-Kolmogorov equations (see for example [6]).

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